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Forced convection in the thermal entrance region of a circular duct with slug flow and viscous dissipation

ANTONIO BARLETTA and ENZO ZANCHINI

Dipartimento di Ingegneria Energetica, Nucleare e del Controllo Ambientale, Università di Bologna, Viale Risorgimento 2, 40136 Bologna, Italy

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Abstract—Slug flow forced convection in a circular duct is studied. The effect of viscous dissipation is analysed in the thermal entrance region. The temperature field and the local Nusselt number are determined analytically for any prescribed axial distribution of wall heat flux. Three examples are considered: a uniform wall heat flux, a linearly varying wall heat flux and an exponentially varying wall heat flux. In the case of a uniform wall heat flux, it is shown that viscous dissipation reduces the value of the local Nusselt number in the whole duct. In the case of a linearly or exponentially increasing wall heat flux, viscous dissipation affects the local Nusselt number only in the thermal entrance region and becomes negligible in the fully developed region. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

In a previous paper [1], an analysis of the thermally developed forced convection in a circular duct with slug flow has been performed by taking into account the effect of viscous dissipation in the fluid. In ref. [1], a viscous dissipation of energy distributed as a Dirac's delta centered next to the wall is assumed. By this model, it is shown that the fully developed value of the Nusselt number is zero for any axial distribution of wall heat flux which tends to zero, when the distance x from the inlet section tends to infinity. Moreover, it is proved that the fully developed value of the Nusselt number depends only on the limits of the wall heat flux $q_w(x)$ and of $[1/q_w(x)][dq_w(x)/dx]$ for $x \to +\infty$. The aim of this paper is to extend the analysis performed in ref. [1] by evaluating the temperature field and the local Nusselt number in the thermal entrance region for an arbitrary axial distribution of wall heat flux.

In the literature, slug flow forced convection in ducts has been studied with reference either to turbulent flow, or to laminar flow in the hydrodynamic entrance region of a Newtonian fluid with a very small value of the Prandtl number, or to the fully developed laminar flow of a pseudoplastic fluid with a negligible power-law index. Moreover, slug flow forced convection can describe a solid rod moving, with good thermal contact, through a heated sleeve. Some of the most important results found in the literature have been reviewed by Shah and Bhatti [2]. For instance, the thermal entrance region has been studied by Golos [3] and by Tyagi and Nigam [4]. Both in ref. [3] and in ref. [4] a boundary condition of convective heat transfer with an external isothermal fluid is considered and approximate analytical solutions are found either by the principle of restricted variation [3] or by Galerkin's technique [4]. The effect of axial heat conduction both in the wall and in the fluid has been taken into account by Soliman [5]. However, the effect of viscous dissipation has been always considered as negligible by these authors. On the other hand, viscous dissipation can not only modify the fully developed value of the Nusselt number, as it has been shown in ref. [1], but can be very important in the thermal entrance region for almost all axial distributions of wall heat flux, as it is shown in the present paper.

MATHEMATICAL MODEL

In this section, the boundary value problem which describes slug flow forced convection with viscous dissipation in a circular duct is outlined. Then, the problem is written in a dimensionless form and relevant dimensionless parameters are determined.

Let us consider slug flow within a circular duct with radius r_0. The axial component of the fluid velocity is uniform with value u_0 within the duct and is zero at the wall. The thermal conductivity and the thermal diffusivity of the fluid are considered to be independent of temperature. The axial heat conduction in the fluid and in the wall are considered as negligible. Since the effect of viscosity is restricted to an infinitesimal layer adjacent to the wall, the power generated per unit volume by viscous dissipation can be expressed by a Dirac's delta distribution centered next to the wall, i.e.

$$\mu\Phi(r) = \phi_0\delta(r-r_0). \quad (1)$$

NOMENCLATURE

a, b arbitrary real numbers employed in equation (A9)
A dimensionless function of ξ and ω defined by equation (26)
a_0, a_n dimensionless coefficients defined by equation (A2)
Br Brinkman number, defined by equation (7)
F dimensionless function of x which appears in equation (3)
g arbitrary dimensionless function of η employed in the Appendix
i $=\sqrt{-1}$, imaginary unit
I_ν modified Bessel function of first kind and order ν
J_ν Bessel function of first kind and order ν
k thermal conductivity of the fluid [W (m^{-1} K^{-1})]
L^*_{th} dimensionless thermal entry length
n arbitrary non-negative integer number
$Nu, Nu(\xi)$ $= 2r_0q_w/[k(T_w - T_b)]$, Nusselt number; local Nusselt number
p dimensionless axial coordinate in the Laplace transformed domain
Pe Peclet number, defined by equation (7)
q_w wall heat flux [W m^{-2}]
q_0 uniform value of the wall heat flux [W m^{-2}]
r radial coordinate [m]
r_0 radius of the duct [m]
Re real part of a complex number
$Res(;)$ residue of a function at a pole
T temperature [K]
T_0 inlet temperature [K]
u_0 uniform value of the axial component of the fluid velocity [m s^{-1}]
x axial coordinate [m]
X arbitary function of r and x employed in equation (30)
y arbitrary real variable.

Greek symbols
α thermal diffusivity of the fluid [m^2 s^{-1}]
β dimensionless coefficient employed for exponentially varying wall heat fluxes
γ dimensionless constant employed in equation (17)
δ Dirac's delta distribution
η dimensionless radial coordinate defined by equation (7)
ϑ dimensionless temperature defined by equation (7)
ϑ_0 function of η and ξ defined by equations (15) and (17)
λ_n nth root of the equation $J_1(y) = 0$
μ dynamic viscosity coefficient [Pa s]
ξ dimensionless axial coordinate defined by equation (7)
ρ mass density [kg m^{-3}]
σ dimensionless function of ξ and β defined by equation (52)
φ dimensionless function of ξ defined by equation (37)
ϕ_0 constant employed in equation (1) [W m^{-1}]
Φ viscous dissipation function [s^{-2}]
χ dimensionless function of ξ defined by equation (44)
Ψ dimensionless function of ξ defined by equation (7)
ω real variable employed in equation (26).

Superscript and subscripts
$\tilde{}$ Laplace transformed function
$'$ dummy integration variable
b bulk value, defined by equation (30)
w value at the wall.

In equation (1), ϕ_0 is a constant which depends on the radius of the duct and on the thermodynamic state of the fluid, while $\delta(r-r_0)$ is Dirac's delta distribution and is such that the integral of $2\pi r\delta(r-r_0)$ in the interval $[0, r_0]$ is equal to one. If the axial distribution of wall heat flux is prescribed and a uniform inlet temperature distribution is assumed, the temperature field is determined by the boundary value problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{u_0}{\alpha}\frac{\partial T}{\partial x} - \frac{\phi_0}{k}\delta(r-r_0) \tag{2}$$

$$k\left.\frac{\partial T}{\partial r}\right|_{r\to r_0} = q_0F(x); \quad \left.\frac{\partial T}{\partial r}\right|_{r\to 0} = 0 \tag{3}$$

$$T(r, 0) = T_0. \tag{4}$$

An equivalent mathematical representation of equations (2) and (3), given in ref. [1], is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{u_0}{\alpha}\frac{\partial T}{\partial x} \tag{5}$$

$$k\left.\frac{\partial T}{\partial r}\right|_{r\to r_0} = q_0F(x) + \frac{\phi_0}{2\pi r_0}; \quad \left.\frac{\partial T}{\partial r}\right|_{r\to 0} = 0. \tag{6}$$

Equations (4)–(6) can be written in a dimensionless form by employing the dimensionless quantities

$$\vartheta = k\frac{T-T_0}{q_0 r_0};\quad \eta = \frac{r}{r_0};\quad \xi = \frac{x}{2r_0 Pe};\quad Pe = \frac{2u_0 r_0}{\alpha}$$

$$Br = \frac{\phi_0}{2\pi r_0 q_0};\quad \Psi(\xi) = F(2r_0 Pe\xi). \tag{7}$$

Equations (4)–(7) yield

$$\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial\vartheta}{\partial\eta}\right) = \frac{1}{4}\frac{\partial\vartheta}{\partial\xi} \tag{8}$$

$$\left.\frac{\partial\vartheta}{\partial\eta}\right|_{\eta\to 1} = \Psi(\xi)+Br;\quad \left.\frac{\partial\vartheta}{\partial\eta}\right|_{\eta\to 0} = 0 \tag{9}$$

$$\vartheta(\eta,0) = 0. \tag{10}$$

Equations (8)–(10) show that the dimensionless temperature depends on the dimensionless parameter Br, as well as on any dimensionless parameter which may appear in function $\Psi(\xi)$.

EVALUATION OF THE TEMPERATURE FIELD

In this section, equations (8)–(10) are solved by the Laplace transform method for an arbitrary function $\Psi(\xi)$. The Laplace transform of the dimensionless temperature field with respect to the dimensionless coordinate ξ is given by

$$\tilde{\vartheta}(\eta,p) = \int_0^{+\infty} e^{-p\xi}\vartheta(\eta,\xi)\,d\xi. \tag{11}$$

On account of the properties of Laplace transforms [6], equations (8), (10) and (11) yield

$$\eta\frac{\partial^2\tilde{\vartheta}}{\partial\eta^2}+\frac{\partial\tilde{\vartheta}}{\partial\eta}-\eta\frac{p}{4}\tilde{\vartheta} = 0 \tag{12}$$

while equation (9) yields

$$\left.\frac{\partial\tilde{\vartheta}}{\partial\eta}\right|_{\eta\to 1} = \tilde{\Psi}(p)+\frac{Br}{p};\quad \left.\frac{\partial\tilde{\vartheta}}{\partial\eta}\right|_{\eta\to 0} = 0. \tag{13}$$

On account of the properties of modified Bessel functions [7], the solution of equations (12) and (13) can be expressed as

$$\tilde{\vartheta}(\eta,p) = \left[\tilde{\Psi}(p)+\frac{Br}{p}\right]\frac{I_0\left(\frac{\sqrt{p}}{2}\eta\right)}{\frac{\sqrt{p}}{2}I_1\left(\frac{\sqrt{p}}{2}\right)}. \tag{14}$$

If one denotes by $\tilde{\vartheta}_0(\eta,p)$ the function

$$\tilde{\vartheta}_0(\eta,p) = \frac{I_0\left(\frac{\sqrt{p}}{2}\eta\right)}{\frac{\sqrt{p}}{2}I_1\left(\frac{\sqrt{p}}{2}\right)} \tag{15}$$

on account of the convolution theorem for Laplace transforms [6], equation (14) yields

$$\vartheta(\eta,\xi) = \int_0^{\xi}[\Psi(\xi')+Br]\vartheta_0(\eta,\xi-\xi')\,d\xi'. \tag{16}$$

Function $\vartheta_0(\eta,\xi)$ can be evaluated by the inversion formula for Laplace transforms [6]

$$\vartheta_0(\eta,\xi) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{p\xi}\tilde{\vartheta}_0(\eta,p)\,dp \tag{17}$$

where γ is any real number such that all singularities of the function $\tilde{\vartheta}_0(\eta,p)$ lie in the complex p plane to the left of the line $Re\{p\} = \gamma$. Although $\tilde{\vartheta}_0(\eta,p)$ has no branch point, equation (15) reveals that this function has infinite simple poles for $p = -4\lambda_n^2$ where, as a consequence of the identity [7]

$$I_1\left(\frac{\sqrt{p}}{2}\right) = -iJ_1\left(i\frac{\sqrt{p}}{2}\right) \tag{18}$$

$\{\lambda_n\}$ is the sequence of roots of the equation $J_1(y) = 0$. The lower root of this equation is $y = \lambda_0 = 0$. The integral on the right hand side of equation (17) can be evaluated by a contour integration of $e^{p\xi}\tilde{\vartheta}_0(\eta,p)$ on a semicircular closed path which lies to the left of the line $Re\{p\} = \gamma$ and centered at $p = \gamma$ [6]. Then, one lets the radius of the semicircular path tend to infinity and on account of Cauchy's residue theorem, equation (17) can be rewritten as

$$\vartheta_0(\eta,\xi) = \sum_{n=0}^{\infty} Res(e^{p\xi}\tilde{\vartheta}_0(\eta,p);\quad p = -4\lambda_n^2). \tag{19}$$

As a consequence of the properties of Bessel functions [7], the residues which appear in equation (19) can be expressed, for $n = 0$, as

$$Res(e^{p\eta}\tilde{\vartheta}_0(\xi,p);\quad p = -4\lambda_0^2) = 8 \tag{20}$$

while, for $n > 0$, as

$$Res(e^{p\eta}\tilde{\vartheta}_0(\xi,p);\quad p = -4\lambda_n^2) = 16e^{-4\lambda_n^2\xi}\frac{J_0(\lambda_n\eta)}{J_0(\lambda_n)-J_2(\lambda_n)}. \tag{21}$$

On account of the identity [7]

$$J_2(y) = \frac{2}{y}J_1(y)-J_0(y) \tag{22}$$

one obtains $J_2(\lambda_n) = -J_0(\lambda_n)$, so that equation (21) can be rewritten as

$$Res(e^{p\eta}\tilde{\vartheta}_0(\xi,p);\quad p = -4\lambda_n^2) = 8e^{-4\lambda_n^2\xi}\frac{J_0(\lambda_n\eta)}{J_0(\lambda_n)}. \tag{23}$$

Equations (19), (20) and (23) yield

$$\vartheta_0(\eta,\xi) = 8\left[1+\sum_{n=1}^{\infty} e^{-4\lambda_n^2\xi}\frac{J_0(\lambda_n\eta)}{J_0(\lambda_n)}\right]. \tag{24}$$

By substituting equation (24) in equation (16), one obtains

$$\vartheta(\eta,\xi) = 8[A(\xi,0)+\xi Br] + 8\sum_{n=1}^{\infty}\left[A(\xi,4\lambda_n^2)+\frac{Br}{4\lambda_n^2}(1-e^{-4\lambda_n^2\xi})\right]\frac{J_0(\lambda_n\eta)}{J_0(\lambda_n)} \quad (25)$$

where function $A(\xi,\omega)$ is defined as

$$A(\xi,\omega) = e^{-\omega\xi}\int_0^{\xi} e^{\omega\xi'}\Psi(\xi')\,d\xi'. \quad (26)$$

As it is proved in the Appendix, the identity

$$\eta^2-\frac{1}{2} = 4\sum_{n=1}^{\infty}\frac{J_0(\lambda_n\eta)}{\lambda_n^2 J_0(\lambda_n)} \quad (27)$$

holds, so that equation (25) can be rewritten as

$$\vartheta(\eta,\xi) = 8[A(\xi,0)+\xi Br]+\frac{Br}{2}\left(\eta^2-\frac{1}{2}\right) + 8\sum_{n=1}^{\infty}\left[A(\xi,4\lambda_n^2)-\frac{Br}{4\lambda_n^2}e^{-4\lambda_n^2\xi}\right]\frac{J_0(\lambda_n\eta)}{J_0(\lambda_n)}. \quad (28)$$

THE LOCAL NUSSELT NUMBER

In this section, equation (28) is employed to determine the local Nusselt number for an arbitrary function $\Psi(\xi)$. On account of equations (3) and (7), the local Nusselt number can be evaluated as

$$Nu(\xi) = \frac{2r_0}{k}\frac{q_0F(x)}{T_w(x)-T_b(x)} = \frac{2\Psi(\xi)}{\vartheta_w(\xi)-\vartheta_b(\xi)} \quad (29)$$

where the bulk value of an arbitrary function $X(r,x)$ is defined as

$$X_b(x) = \frac{2}{r_0^2}\int_0^{r_0} X(r',x)r'\,dr'. \quad (30)$$

On account of equations (9) and (30), an integration of both sides of equation (8) with respect to η in the interval [0, 1] yields

$$\frac{d\vartheta_b(\xi)}{d\xi} = 8[\Psi(\xi)+Br]. \quad (31)$$

Since equations (10) and (30) ensure that $\vartheta_b(0) = 0$, by integrating equation (31) and by employing equation (26), one obtains

$$\vartheta_b(\xi) = 8[A(\xi,0)+\xi Br]. \quad (32)$$

Then, on account of equations (28), (29) and (32), the local Nusselt number can be expressed as

$$Nu(\xi) = \frac{8\Psi(\xi)}{Br+32\sum_{n=1}^{\infty}\left[A(\xi,4\lambda_n^2)-\frac{Br}{4\lambda_n^2}e^{-4\lambda_n^2\xi}\right]}. \quad (33)$$

UNIFORM WALL HEAT FLUX

In this section, equations (28) and (33) are employed in the case of a uniform wall heat flux, i.e. in the case $\Psi(\xi) = 1$. If the wall heat flux is uniform, equation (26) yields

$$A(\xi,\omega) = \begin{cases} \xi, & \omega = 0 \\ \dfrac{1-e^{-\omega\xi}}{\omega}, & \omega \neq 0. \end{cases} \quad (34)$$

On account of equations (27), (28) and (34), $\vartheta(\eta,\xi)$ is given by

$$\vartheta(\eta,\xi) = 8(1+Br)\times\left[\xi+\frac{1}{16}\left(\eta^2-\frac{1}{2}\right)-\frac{1}{4}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n\eta)}{\lambda_n^2J_0(\lambda_n)}e^{-4\lambda_n^2\xi}\right]. \quad (35)$$

Moreover, equations (27) and (33) yield

$$Nu(\xi) = \frac{8}{(1+Br)\left(1-8\sum_{n=1}^{\infty}\frac{e^{-4\lambda_n^2\xi}}{\lambda_n^2}\right)}. \quad (36)$$

In the case $Br = 0$, equations (35) and (36) coincide with the corresponding expressions of $\vartheta(\eta,\xi)$ and of $Nu(\xi)$ reported in Shah and Bhatti [2]. Let us define function $\varphi(\xi)$ as

$$\varphi(\xi) = \sum_{n=1}^{\infty}\frac{e^{-4\lambda_n^2\xi}}{\lambda_n^2}. \quad (37)$$

Values of function $\varphi(\xi)$ are reported in Table 1. On account of equation (37), equation (36) can be rewritten as

$$Nu(\xi) = \frac{8}{(1+Br)[1-8\varphi(\xi)]}. \quad (38)$$

Equations (27) and (37) ensure that $\varphi(0) = 1/8$, so that the local Nusselt number becomes singular at the inlet section for any value of Br. Moreover, in the limit $\xi \to +\infty$, $\varphi(\xi)$ tends to zero and equation (38) reveals that the fully developed value of the Nusselt number is $8/(1+Br)$ as found in ref. [1]. Equation (38) shows that, if $q_0 > 0$, the local Nusselt number is a decreasing function of Br. On the other hand, if $Br = -1$, for every value of ξ the local Nusselt number is singular and, as a consequence of equation (35), $\vartheta(\eta,\xi) = 0$ at every point. Indeed, when $Br = -1$, the wall is cooled so that all the heat generated by viscous dissipation is subtracted to the fluid. Therefore, for every value of x, the fluid temperature profile is the same as at the inlet section, i.e. it is uniform with value T_0.

In Fig. 1, three plots of $Nu(\xi)$ for $Br = 0$, $Br = 1$ and $Br = 10$ are reported. This figure shows that the effect of viscous dissipation in the thermal entrance region is relevant especially for low values of ξ. As usual, the dimensionless thermal entry length L_{th}^* is defined as the value of ξ such that $Nu(\xi)$ is equal to

Table 1. Values of $\varphi(\xi)$, $\chi(\xi)$, $\sigma(\xi, 10)$ and $\sigma(\xi, -10)$

ξ	$\varphi(\xi)\times 10$	$\chi(\xi)\times 1000$	$\sigma(\xi,10)\times 100$	$\sigma(\xi,-10)\times 100$
0.00000	1.25000	5.20833	5.25345	8.16420
0.00001	1.21462	5.20343	5.07757	7.98568
0.00005	1.17171	5.18438	4.86699	7.76484
0.00010	1.14015	5.16128	4.71399	7.59938
0.00050	1.01256	4.99049	4.11097	6.90376
0.00100	0.92280	4.79755	3.70111	6.38868
0.00500	0.59770	3.62565	2.30546	4.34597
0.01000	0.40854	2.63773	1.54838	3.03669
0.02000	0.21442	1.44134	0.80297	1.61770
0.03000	0.11752	0.79779	0.43877	0.88993
0.04000	0.06509	0.44297	0.24284	0.49337
0.05000	0.03615	0.24616	0.13483	0.27406
0.06000	0.02009	0.13681	0.07493	0.15231
0.07000	0.01117	0.07604	0.04164	0.08465
0.08000	0.00621	0.04227	0.02315	0.04705
0.09000	0.00345	0.02349	0.01287	0.02615
0.10000	0.00192	0.01306	0.00715	0.01454
0.50000	0.00000	0.00000	0.00000	0.00000
∞	0.00000	0.00000	0.00000	0.00000

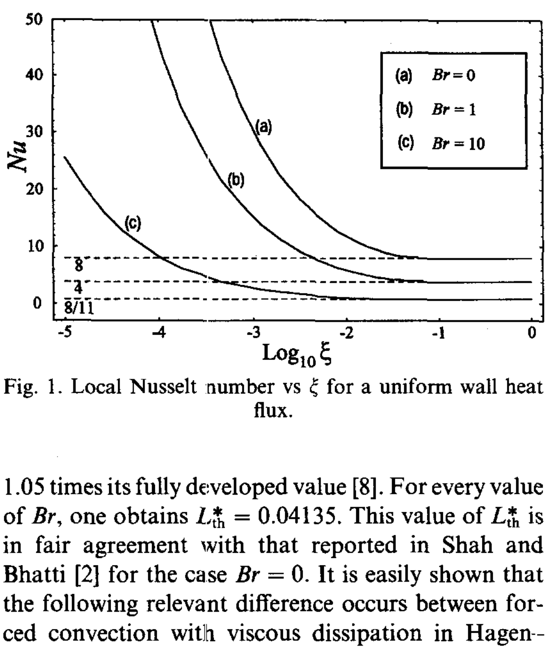

Fig. 1. Local Nusselt number vs ξ for a uniform wall heat flux.

1.05 times its fully developed value [8]. For every value of Br, one obtains $L^*_{th} = 0.04135$. This value of L^*_{th} is in fair agreement with that reported in Shah and Bhatti [2] for the case $Br = 0$. It is easily shown that the following relevant difference occurs between forced convection with viscous dissipation in Hagen–Pouiseuille flow and forced convection with viscous dissipation in slug flow. Ou and Cheng [9] have shown that in the thermal entrance region of the forced convection with viscous dissipation and Hagen–Pouiseuille flow there exists, in the case of uniform wall heat flux with $Br < -11/48$, an axial position where the local Nusselt number becomes singular. Such singularities do not occur in the case of slug flow. In fact, if $Br \neq -1$, equation (38) ensures that a singularity of $Nu(\xi)$ can occur only for a value of ξ such that $\varphi(\xi) = 1/8$. Table 1 shows that the only value of ξ such that $\varphi(\xi) = 1/8$ is $\xi = 0$.

LINEARLY VARYING WALL HEAT FLUX

In this section, equations (28) and (33) are employed in the case of a linearly varying wall heat flux, i.e. in the case $\Psi(\xi) = \xi$. If the wall heat flux is linearly varying, equation (26) yields

$$A(\xi,\omega) = \begin{cases} \dfrac{\xi^2}{2}, & \omega = 0 \\ \dfrac{e^{-\omega\xi}+\omega\xi-1}{\omega^2}, & \omega \neq 0. \end{cases} \quad (39)$$

On account of equations (27), (28) and (39), $\vartheta(\eta,\xi)$ is given by

$$\vartheta(\eta,\xi) = 8\left(\frac{\xi^2}{2}+\xi Br\right)+\frac{(Br+\xi)}{2}\left(\eta^2-\frac{1}{2}\right)$$
$$+2\sum_{n=1}^{\infty}\left[\left(\frac{1}{4\lambda_n^2}-Br\right)e^{-4\lambda_n^2\xi}-\frac{1}{4\lambda_n^2}\right]\frac{J_0(\lambda_n\eta)}{\lambda_n^2 J_0(\lambda_n)}. \quad (40)$$

As it is proved in the Appendix, the identity

$$\eta^4-2\eta^2+\frac{2}{3} = -64\sum_{n=1}^{\infty}\frac{J_0(\lambda_n\eta)}{\lambda_n^4 J_0(\lambda_n)} \quad (41)$$

holds, so that equation (40) can be rewritten as

$$\vartheta(\eta,\xi) = 4\xi(\xi+2Br)+\frac{(\xi+Br)}{2}\left(\eta^2-\frac{1}{2}\right)$$
$$+\frac{1}{128}\left(\eta^4-2\eta^2+\frac{2}{3}\right)+2\sum_{n=1}^{\infty}\left(\frac{1}{4\lambda_n^2}-Br\right)$$
$$\times\frac{J_0(\lambda_n\eta)}{\lambda_n^2 J_0(\lambda_n)}e^{-4\lambda_n^2\xi}. \quad (42)$$

Moreover, equations (27), (33) and (41) yield

$$Nu(\xi) = \frac{8\xi}{\xi+Br-\dfrac{1}{96}+8\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{4\lambda_n^2}-Br\right)\frac{e^{-4\lambda_n^2\xi}}{\lambda_n^2}}. \quad (43)$$

Let us define function $\chi(\xi)$ as

$$\chi(\xi) = \sum_{n=1}^{\infty} \frac{e^{-4\lambda_n^2 \xi}}{\lambda_n^4}. \tag{44}$$

Values of function $\chi(\xi)$ are reported in Table 1. On account of equations (37) and (44), equation (43) can be rewritten as

$$Nu(\xi) = \frac{8\xi}{\xi + Br - \frac{1}{96} + 2\chi(\xi) - 8Br\varphi(\xi)}. \tag{45}$$

Equations (41) and (44) ensure that $\chi(0) = 1/192$. Therefore, by employing equations (37), (44) and (45), it can be easily proved that, for $Br = 0$, the local Nusselt number becomes singular at the inlet section, while, for $Br \neq 0$, the local Nusselt number tends to zero at the inlet section. Moreover, in the limit $\xi \to +\infty$, $\varphi(\xi)$ tends to zero and equation (45) reveals that the fully developed value of the Nusselt number is 8 as found in ref. [1].

In Fig. 2, three plots of $Nu(\xi)$ for $Br = 0.1$, $Br = 1$ and $Br = 10$ are reported. This figure shows how the thermal entrance region increases its length as Br increases. In Fig. 3, the behaviour of $Nu(\xi)$ for negative values of Br is represented. A relevant feature of the plots reported in Fig. 3 is that $Nu(\xi)$ presents a singularity whose position depends on the value of Br. For $Br = -0.1$, the local Nusselt number is singular at $\xi = 0.1104$; for $Br = -1$ the singularity is at $\xi = 1.0104$, while, for $Br = -10$, $Nu(\xi)$ presents a singularity at $\xi = 10.0104$. As it can be inferred from the values reported in Table 1 and from equation (45), for $Br \leqslant -0.5$, the position of the singularity is given, with excellent accuracy, by $\xi = -Br + 1/96$. In Fig. 4, three plots of $Nu(\xi)$ for $Br = 0$, $Br = 10^{-4}$ and $Br = 10^{-3}$ are reported. These plots show that even a very small value of Br yields a thermal entrance region different from that which corresponds to $Br = 0$. In particular, for $Br = 10^{-3}$ and for $Br = 10^{-4}$, the local Nusselt number reaches a maximum at $\xi = 1.7905 \times 10^{-3}$ and at $\xi = 1.5849 \times 10^{-4}$, respectively. The dimensionless thermal entry length, in the case $Br = 0$, is given by $L_{th}^* = 0.21881$.

In Fig. 5, plots of $\vartheta - \vartheta_w$ vs η for $Br = -1$ and for $\xi = 0.5$, $\xi = 1.0104$ and $\xi = 1.5$ are presented. This figure reveals that at the axial position $\xi = 1.0104$, which corresponds to a singularity of $Nu(\xi)$, the temperature is almost uniform, while the slope of the temperature profile at $r = r_0$ undergoes a change of sign. Indeed, when $Nu(\xi)$ becomes singular, the bulk value of $\vartheta - \vartheta_w$ vanishes. The circumstance is somewhat similar to that which is pointed out by Ou and Cheng [9] in the case of a Hagen–Pouiseuille flow with uniform wall heat flux. These authors point out that such singularities occur when there exists an axial position where $T_w = T_b$. By moving in the axial direction and crossing the position where $T_w = T_b$, the sign of $T_w - T_b$ changes, while that of q_w remains unchanged. As a consequence, the definition of Nusselt number implies that $Nu(\xi)$ also undergoes a sign

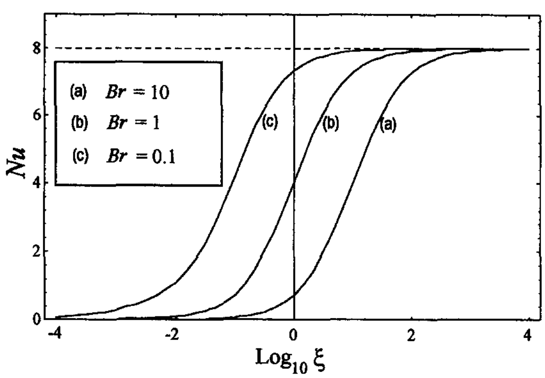

Fig. 2. Local Nusselt number vs ξ for a linearly varying wall heat flux.

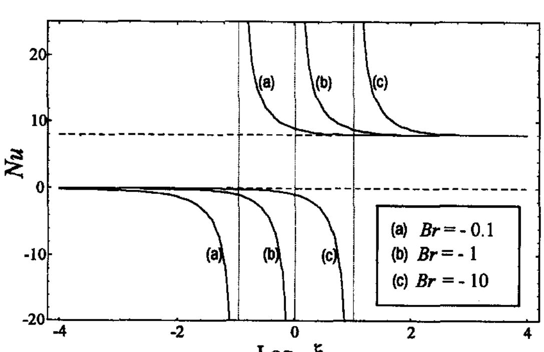

Fig. 3. Local Nusselt number vs ξ for a linearly varying wall heat flux.

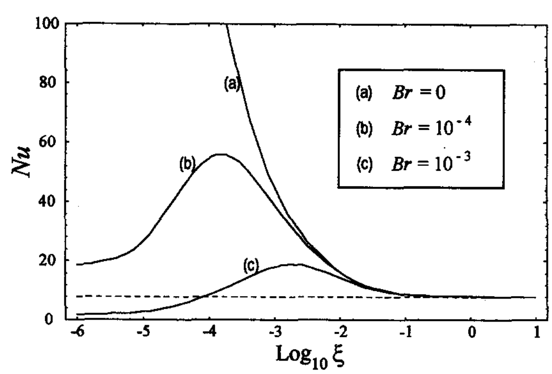

Fig. 4. Local Nusselt number vs ξ for a linearly varying wall heat flux.

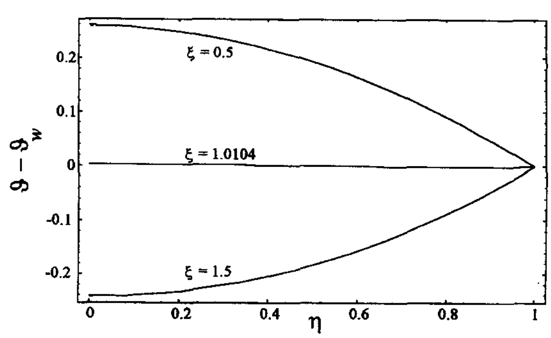

Fig. 5. $\vartheta - \vartheta_w$ vs η for a linearly varying wall heat flux and $Br = -1$.

change and, at the position where $T_w = T_b$, becomes singular. Singular points of $Nu(\xi)$ have also been revealed in the case of uniform wall temperature by Ou and Cheng [10], for a Hagen–Pouiseuille flow with viscous dissipation and by Lawal and Mujumdar [11] for the laminar flow of a power-law fluid with viscous dissipation. Moreover, singularities of the local Nusselt number have been revealed by Lin *et al.* [12] for a Hagen–Pouiseuille flow with viscous dissipation and external convection with an isothermal fluid.

EXPONENTIALLY VARYING WALL HEAT FLUX

In this section, the dimensionless temperature field and the local Nusselt number are analysed in the case of an exponentially varying wall heat flux, i.e. in the case $\Psi(\xi) = e^{2\beta\xi}$. If the wall heat flux is exponentially varying, equation (26) yields

$$A(\xi,\omega) = \frac{e^{2\beta\xi} - e^{-\omega\xi}}{\omega + 2\beta}. \quad (46)$$

On account of equations (28) and (46), $\vartheta(\eta,\xi)$ is given by

$$\vartheta(\eta,\xi) = 8\left[\frac{e^{2\beta\xi}-1}{2\beta} + \xi Br\right] + \frac{Br}{2}\left(\eta^2 - \frac{1}{2}\right) + 4\sum_{n=1}^{\infty}\left[\frac{e^{2\beta\xi}}{2\lambda_n^2+\beta} - \left(\frac{Br}{2\lambda_n^2} + \frac{1}{2\lambda_n^2+\beta}\right)e^{-4\lambda_n^2\xi}\right] \times \frac{J_0(\lambda_n\eta)}{J_0(\lambda_n)}. \quad (47)$$

As is proved in the Appendix, the identity

$$I_0\left(\eta\sqrt{\frac{\beta}{2}}\right) - 2\sqrt{\frac{2}{\beta}}I_1\left(\sqrt{\frac{\beta}{2}}\right) = 2\sqrt{2\beta}\,I_1\left(\sqrt{\frac{\beta}{2}}\right)\sum_{n=1}^{\infty}\frac{J_0(\lambda_n\eta)}{(2\lambda_n^2+\beta)J_0(\lambda_n)} \quad (48)$$

holds, so that equation (47) can be rewritten as

$$\vartheta(\eta,\xi) = 8\left[\frac{e^{2\beta\xi}-1}{2\beta} + \xi Br\right] + \frac{Br}{2}\left(\eta^2 - \frac{1}{2}\right) + \sqrt{\frac{2}{\beta}}\left[\frac{I_0\left(\eta\sqrt{\frac{\beta}{2}}\right)}{I_1\left(\sqrt{\frac{\beta}{2}}\right)} - 2\sqrt{\frac{2}{\beta}}\right]e^{2\beta\xi} - 4\sum_{n=1}^{\infty}\left(\frac{Br}{2\lambda_n^2} + \frac{1}{2\lambda_n^2+\beta}\right)\frac{J_0(\lambda_n\eta)}{J_0(\lambda_n)}e^{-4\lambda_n^2\xi}. \quad (49)$$

On account of the identity [7]

$$I_2(y) = I_0(y) - \frac{2}{y}I_1(y) \quad (50)$$

and of equations (27), (33), (49), one obtains

$$Nu(\xi) = 8e^{2\beta\xi}\left[Br + 4\sqrt{\frac{2}{\beta}}\frac{I_2\left(\sqrt{\frac{\beta}{2}}\right)}{I_1\left(\sqrt{\frac{\beta}{2}}\right)}e^{2\beta\xi} - 16\sum_{n=1}^{\infty}\left(\frac{Br}{2\lambda_n^2} + \frac{1}{2\lambda_n^2+\beta}\right)e^{-4\lambda_n^2\xi}\right]^{-1}. \quad (51)$$

Let us define function $\sigma(\xi,\beta)$ as

$$\sigma(\xi,\beta) = \sum_{n=1}^{\infty}\frac{e^{-4\lambda_n^2\xi}}{2\lambda_n^2+\beta}. \quad (52)$$

Values of the functions $\sigma(\xi, 10)$ and $\sigma(\xi, -10)$ are reported in Table 1. On account of equations (37) and (52), equation (51) can be rewritten as

$$Nu(\xi) = 8e^{2\beta\xi}\left[Br + 4\sqrt{\frac{2}{\beta}}\frac{I_2\left(\sqrt{\frac{\beta}{2}}\right)}{I_1\left(\sqrt{\frac{\beta}{2}}\right)}e^{2\beta\xi} - 8Br\varphi(\xi) - 16\sigma(\xi,\beta)\right]^{-1}. \quad (53)$$

On account of equations (48), (50) and (52), one obtains

$$\sigma(0,\beta) = \frac{1}{2\sqrt{2\beta}}\frac{I_2\left(\sqrt{\frac{\beta}{2}}\right)}{I_1\left(\sqrt{\frac{\beta}{2}}\right)}. \quad (54)$$

Since $\varphi(0) = 1/8$, as a consequence of equation (54), the local Nusselt number becomes singular at the inlet section for any value of Br. Moreover, in the limit $\xi \to +\infty$, both $\varphi(\xi)$ and $\sigma(\xi,\beta)$ tend to zero and equation (53) reveals that, if $\beta > 0$, the fully developed value of the Nusselt number is given by

$$\lim_{\xi\to+\infty} Nu(\xi) = \sqrt{2\beta}\frac{I_1\left(\sqrt{\frac{\beta}{2}}\right)}{I_2\left(\sqrt{\frac{\beta}{2}}\right)}. \quad (55)$$

as found in ref. [1]. On the other hand, if $\beta < 0$, equation (53) ensures that the fully developed value of the Nusselt number is zero both in the case $Br \neq 0$ and in the case $Br = 0$ with $\beta \leqslant 2\lambda_1^2 = -29.364$. Finally, if $Br = 0$ and $-2\lambda_1^2 < \beta < 0$, the fully developed value of the Nusselt number is given by

$$\lim_{\xi \to +\infty} Nu(\xi) = \sqrt{2|\beta|}\,\frac{J_1\left(\sqrt{\frac{|\beta|}{2}}\right)}{J_2\left(\sqrt{\frac{|\beta|}{2}}\right)}. \tag{56}$$

However, it should be noted that, if $Br = 0$ and $\beta \leqslant -2\lambda_1^2$, the fully developed value of $Nu(\xi)$ depends on the choice of the inlet temperature profile. If the dimensionless inlet temperature profile is uniform, the fully developed value of $Nu(\xi)$ is zero, as it has been pointed out above. On the other hand, as is proved in ref. [1], if the dimensionless inlet temperature profile is given by

$$\vartheta(\eta, 0) = \frac{4}{|\beta|} + \sqrt{\frac{2}{|\beta|}}\,\frac{J_0\left(\eta\sqrt{\frac{|\beta|}{2}}\right)}{J_1\left(\sqrt{\frac{|\beta|}{2}}\right)} \tag{57}$$

the fully developed value of $Nu(\xi)$ is still given by equation (56). However, in ref. [1] it has been pointed out that, for every wall heat flux which tends to zero for $\xi \to +\infty$, viscous dissipation cannot be neglected in the thermally developed region. Therefore, in the case of an exponentially varying wall heat flux with $Br = 0$ and $\beta \leqslant -2\lambda_1^2$, the dependence of the fully developed value of $Nu(\xi)$ on the inlet temperature profile has no physical relevance.

In Fig. 6, four plots of $Nu(\xi)$ for $Br = 0$, $Br = 1$, $Br = 10$ and $Br = 100$ are reported in the case $\beta = 10$. If $\beta = 10$, the dimensionless thermal entry length for $Br = 0$ is $L^*_{th} = 0.02935$, while on account of equation (55), the fully developed value of $Nu(\xi)$ is 9.51755. Figure 6 shows that, for $Br = 1$, $Br = 10$ and $Br = 100$, $Nu(\xi)$ is a decreasing function for low values of ξ, then it reaches a minimum and starts to increase towards its fully developed value.

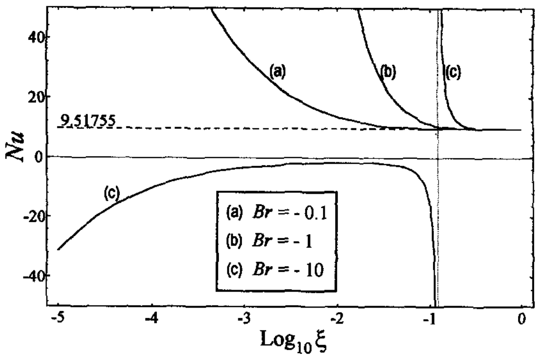

Fig. 7. Local Nusselt number vs ξ for an exponentially varying wall heat flux and $\beta = 10$.

In Fig. 7, the behaviour of $Nu(\xi)$ for $\beta = 10$ and negative values of Br is represented. Figure 7 shows that, for $Br = -10$, $Nu(\xi)$ presents a singularity at $\xi = 0.12385$. It is easily verified that a singularity is present in the thermal entrance region if $Br < -1$. The axial position where $Nu(\xi)$ is singular, is more and more distant from the inlet section as Br increases. Indeed, if $Br < -1$, equation (31) shows that, for $0 < \xi < \ln|Br|/(2\beta)$, $\vartheta_b(\xi)$ is a decreasing function. Since $\vartheta_b(0) = 0$, one can conclude that, for $0 < \xi < \ln|Br|/(2\beta)$, $\vartheta_b(\xi)$ is negative. Then, $\vartheta_b(\xi)$ reaches a minimum at $\xi = \ln|Br|/(2\beta)$ and, for $\xi > \ln|Br|/(2\beta)$, increases. Indeed, Fig. 8 presents the behaviour of $\vartheta_b(\xi)$ and $\vartheta_w(\xi)$ for $\beta = 10$ and $Br = -10$. This figure shows that, for $\xi < 0.12385$, $\vartheta_b(\xi)$ is greater than $\vartheta_w(\xi)$, while the reverse is true for $\xi > 0.12385$. As stated above, $\xi = 0.12385$ is the position where $Nu(\xi)$ is singular. Let us point out that the value of ξ which yields the singularity is slightly greater than that which yields the minimum of $\vartheta_b(\xi)$, i.e. $\xi = (\ln 10)/20 = 0.11513$.

In Fig. 9, three plots of $Nu(\xi)$ for $Br = 0$, $Br = 1$ and $Br = 10$ are reported in the case $\beta = -10$. These plots represent the strong difference between the cases $Br = 0$ and $Br \neq 0$. If $Br = 0$, the fully developed value of $Nu(\xi)$ can be evaluated by equation (56) and is given by 6.12430. If $Br \neq 0$, the fully developed value of $Nu(\xi)$ is zero. In the case $Br = 0$, the dimensionless thermal entry length is $L^*_{th} = 0.06554$.

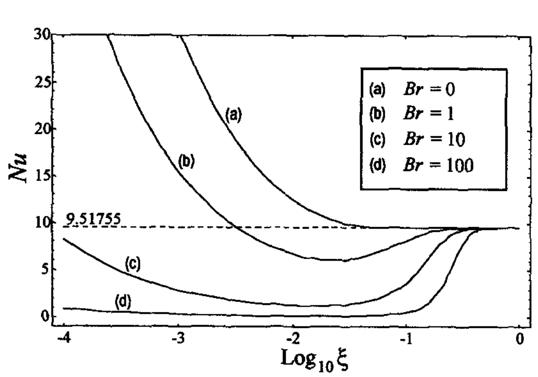

Fig. 6. Local Nusselt number vs ξ for an exponentially varying wall heat flux and $\beta = 10$.

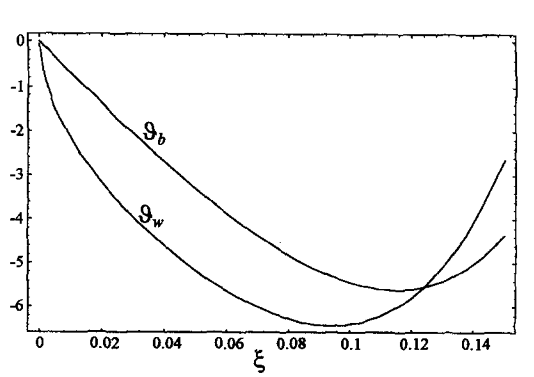

Fig. 8. ϑ_b and ϑ_w vs ξ for an exponentially varying wall heat flux with $\beta = 10$ and $Br = -10$.

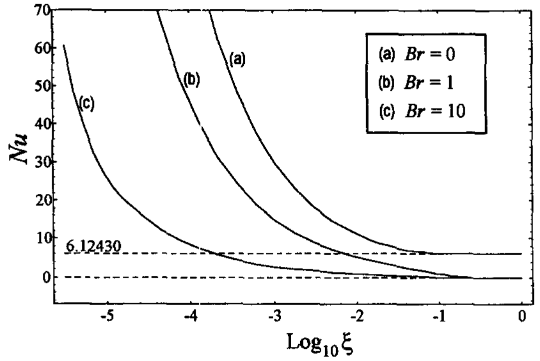

Fig. 9. Local Nusselt number vs ξ for an exponentially varying wall heat flux and $\beta = -10$.

CONCLUSIONS

Slug flow forced convection in the thermal entrance region of a circular duct with an arbitrary axial distribution of wall heat flux has been studied by taking into account the effect of viscous dissipation. The dimensionless temperature field and the local Nusselt number have been evaluated analytically by considering a uniform inlet temperature profile. In particular, three examples have been analysed: uniform wall heat flux, linearly varying wall heat flux and exponentially varying wall heat flux.

For uniform wall heat flux, it has been shown that the local Nusselt number in the thermal entrance region is given by $1/(1+Br)$ times the local Nusselt number in the thermal entrance region of a non-dissipating fluid. Moreover, unlike the case of Hagen–Pouiseuille flow, no singularity of the local Nusselt number occurs in the thermal entrance region for negative values of Br, except for $Br = -1$. Indeed, in the case $Br = -1$ the temperature profile at every axial position is uniform and coincides with the inlet temperature profile, so that the local Nusselt number is singular at any axial position.

For a linearly varying wall heat flux, it has been pointed out that viscous dissipation strongly affects the thermal entrance region even for very small values of Br. For negative values of Br, there exists an axial position where $Nu(\xi)$ is singular. The singularity corresponds to an axial position where $T_w = T_b$ and the wall heat flux is non-vanishing.

For an exponentially increasing wall heat flux with a positive value of Br, viscous dissipation affects the thermal entrance region by producing a minimum of $Nu(\xi)$, so that the fully developed value of the Nusselt number is reached from below. For negative values of Br, singularities of $Nu(\xi)$ occur only if $Br < -1$: the value of ξ which corresponds to a singularity of Nu increases with $|Br|$. As in the case of a linearly varying wall heat flux, this singularity occurs at an axial position where $T_w = T_b$ and the wall heat flux is non-vanishing. If the wall heat flux is exponentially decreasing, a sharp distinction occurs between the case of a non-dissipating fluid and that of a dissipating fluid. In the first case, the fully developed value of $Nu(\xi)$ depends on β, while, in the second case, the fully developed value of $Nu(\xi)$ is zero.

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APPENDIX

A function $g(\eta)$ can be expanded by a series of Bessel functions. In particular, the following relation holds [7]

$$g(\eta) = a_0 + \sum_{n=1}^{\infty} a_n J_0(\lambda_n \eta) \tag{A1}$$

where λ_n is the nth root of $J_1(y) = 0$, while a_0 and a_n are given by

$$a_0 = 2\int_0^1 \eta g(\eta)\,\mathrm{d}\eta\,; \quad a_n = \frac{2}{[J_0(\lambda_n)]^2}\int_0^1 \eta g(\eta) J_0(\lambda_n \eta)\,\mathrm{d}\eta. \tag{A2}$$

Let us consider the expansion of $g(\eta) = \eta^2$. Equation (A2) can be rewritten as

$$a_0 = \frac{1}{2}; \quad a_n = \frac{2}{\lambda_n^4 [J_0(\lambda_n)]^2} \int_0^{\lambda_n} y^3 J_0(y)\, dy. \tag{A3}$$

On account of the identity [7]

$$\int y^3 J_0(y)\, dy = (y^3 - 4y) J_1(y) + 2y^2 J_0(y) \tag{A4}$$

equations (A1) and (A3) yield

$$\eta^2 - \frac{1}{2} = 4 \sum_{n=1}^{\infty} \frac{J_0(\lambda_n \eta)}{\lambda_n^2 J_0(\lambda_n)}. \tag{A5}$$

Let us consider the expansion of $g(\eta) = \eta^4$. Equation (A2) can be rewritten as

$$a_0 = \frac{1}{3}; \quad a_n = \frac{2}{\lambda_n^6 [J_0(\lambda_n)]^2} \int_0^{\lambda_n} y^5 J_0(y)\, dy. \tag{A6}$$

On account of the identity [7]

$$\int y^5 J_0(y)\, dy = (y^5 - 16y^3 + 64y) J_1(y) + 4(y^4 - 8y^2) J_0(y) \tag{A7}$$

equations (A1), (A5) and (A6) yield

$$\eta^4 - 2\eta^2 + \frac{2}{3} = -64 \sum_{n=1}^{\infty} \frac{J_0(\lambda_n \eta)}{\lambda_n^4 J_0(\lambda_n)}. \tag{A8}$$

Finally, let us consider the expansion of $g(\eta) = I_0(\eta\sqrt{\beta/2})$. On account of the identity [7]

$$\int_0^1 \eta J_0(a\eta) I_0(b\eta)\, d\eta = \frac{b J_0(a) I_1(b) + a J_1(a) I_0(b)}{a^2 + b^2} \tag{A9}$$

equation (A2) can be rewritten as

$$a_0 = 2\sqrt{\frac{2}{\beta}} I_1\left(\sqrt{\frac{\beta}{2}}\right); \quad a_n = \frac{2\sqrt{2\beta} I_1\left(\sqrt{\frac{\beta}{2}}\right)}{J_0(\lambda_n)(2\lambda_n^2 + \beta)}. \tag{A10}$$

Equations (A1) and (A10) yield

$$I_0\left(\eta\sqrt{\frac{\beta}{2}}\right) - 2\sqrt{\frac{2}{\beta}} I_1\left(\sqrt{\frac{\beta}{2}}\right) = 2\sqrt{2\beta} I_1\left(\sqrt{\frac{\beta}{2}}\right) \sum_{n=1}^{\infty} \frac{J_0(\lambda_n \eta)}{J_0(\lambda_n)(2\lambda_n^2 + \beta)}. \tag{A11}$$